B.Sc II part Paper IV

Differential Equation (Singular Solution)

Let the diff. Eqn. be denoted by $f(x, y, p) = 0$; $p = \frac{dy}{dx}$ and its general Solution by $\phi(x, y, c) = 0$ Where c is an arbitrary constant. The existence of Singular solution depends upon the p-discriminant of $f(x, y, p) = 0$ and the c-discriminant of $\phi(x, y, c) = 0$. Hence prior to embarking on the definition and existence of Singular Solution, we look at the discriminant of an equation whether ~~algebraic~~ algebraic or differential.

The discriminant of a quadratic eqn $ax^2 + bx + c = 0$ is $b^2 - 4ac$. The vanishing of the discriminant i.e $b^2 - 4ac = 0$ expresses the condition that the two roots of the given eqn. are equal. i.e two values of x are equal. Similarly if $f(x) = 0$ be any eqn. of higher order than two, then as we Know from Theory of Equations that the Condition for equal roots is obtained by eliminating x between $f(x) = 0$ and $f'(x) = 0$.

Let $f(x, y, p) = 0$ be a differential eqn. and let its Solution be $\phi(x, y, c) = 0$ where c is an arbitrary const. Then the c-discriminant is obtained by eliminating c between $\phi(x, y, c) = 0$ and $\frac{\partial \phi}{\partial c} = 0$; treating x, y as constants and its vanishing expresses the condition that the eqn. $\phi(x,y,c)=0$ regarded as an equation in c, has equal values c.

Hence we can say that the c-discriminant of $\phi(x,y,c)=0$ represents the locus for each point of which $\phi(x,y,c)=0$ has equal values of c.

Again, if $f(x,y,p)=0$ be the differential equation whose general solution is $\phi(x,y,c)=0$, then the p-discriminant is obtained by eliminating p between $f(x,y,p)=0$ and $\frac{\partial f}{\partial p}=0$, treating x, y as constants and its vanishing express the condition that the equation $f(x,y,p)=0$ regarded as an equation in p has equal values of p.

Hence, we can say that the p-discriminant of $f(x,y,p)=0$ represents the locus for each point of which $f(x,y,p)=0$ has equal values of p.

If $y = px + f(p)$; $p = \frac{dy}{dx}$ ——— (1)

be a diff. eqn. in Clairaut's form, then its general solution is $y = cx + f(c)$ ——— (2)

where c is an arbitrary constant. Obviously the p-discriminant of (1) and c-discriminant of (2) are the same and that gives us what is called the singular solution of the D.E (1).

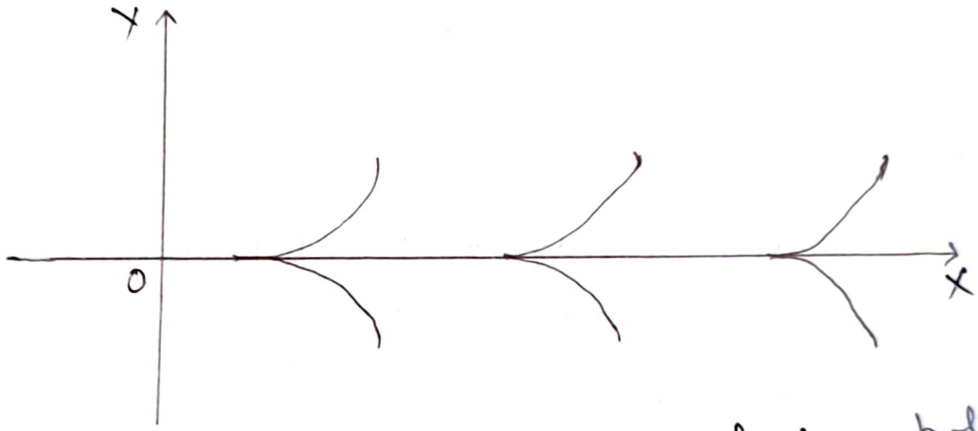
However, if $f(x,y,p)=0$ (3) be any diff. eqn. not necessary in Clairaut's form and if $\phi(x,y,c)=0$ ——— (4) be its solution, then are we suppose that p-discriminant of (3) and c-discriminant of (4) are the same? obviously not.

1. Example - Consider the family of Semi-cubical parabolas $f(x,y,c) = y^2 - (x+c)^3 = 0$.

Soln. Here $\frac{\partial f}{\partial c} = 3(x+c)^2 = 0$.

Eliminating c, we obtain $y^2 = 0$ i.e. $y = 0$.

It can be verified that $(-c, 0)$ is a singular point (a cusp). Thus $y = 0$ is the locus of singular points, c-discriminant not an envelope, even though it is c-discriminant. The locus here is called cusp locus.



Family of semi-cubical parabolas.

[Here the neighbouring curves donot intersect each other]

2. Example - Consider the family of curves

$x^2(x-a) + (x+a)(y-c)^2 = 0$.

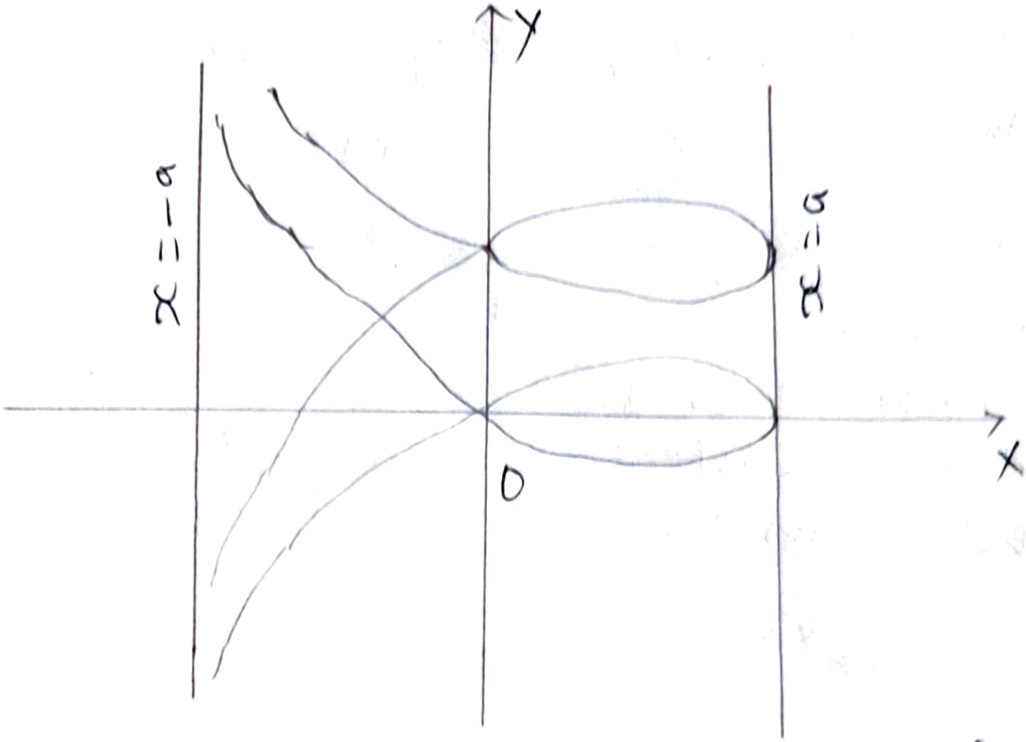
Where a is a constant and c is a parameter.

Soln: Differentiating w.r.t. c, we get

$-2(x+a)(y-c) = 0$.

Eliminating c, we get $x^2(x-a) = 0$.

Thus the c-discriminant consists of two lines $x = 0$ and $x = a$.



If we trace given curve, $(y-c)^2 = \frac{x^2(x-a)}{x+a}$ we shall find that $x = 0$ (y-axis) is the locus of its singular points (cusp locus) and $x = a$ is the tangent to each curve and hence envelope to the given system of curves.

Further at the points, where p's are equal will include the envelope. (ultimate intersection of consecutive curves, the p's $(= \frac{dy}{dx})$ for the intersecting curves become equal and hence the locus of the points)

Hence the envelope is also contained in the p-discriminant.

Hence, the p-discriminant of $f(x, y, p) = 0$ contains the equation to the envelope of $\phi(x, y, c) = 0$, which is the solution of the given differential equation.

Thus the envelope is contained both in the c-discriminant as well as in p-discriminant.